Robust optimization of uncertain multistage inventory systems with inexact data in decision rules

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Overview

1. Intro inexact data
2. Intro robust optimization
3. Robust optimization techniques
4. New methodology
5. Numerical example

Data uncertainty in practical applications

Optimization problems are affected by uncertainty in their parameters due to:

1. Measurement errors

   physical experiments, weather observations, . . .

2. Prediction errors

   future demand, returns, . . .

3. Implementation errors

   optimal temperature, size, . . .

4. System data errors

   inventory records, miscodings, . . .

Robust Optimization (RO) techniques find solutions that are robust against uncertainties in the parameters.

Evidence of poor data quality

Despite developments in our Big Data era poor data quality is still a big issue.

- Redman (1998):
  - $1 - 5\%$ *of data fields are erred.*
- DeHoratius and Raman (2008):
  - *Over 6 out of 10 inventory records are inaccurate.*
- Haug et al. (2011):
  - *Not even half of the companies is very confident in the quality of their data.*

...

Evolution of Robust Optimization

- Early 70s: First note on RO by Soyster.
- Late 90s: Research kicked off due to Ben-Tal, Nemirovski and El Ghaoui.
- 2004: Bertsimas and Sim's budget uncertainty model.
- 2004: Adjustable Robust Optimization by Ben-Tal et al.
- 2009: Book *Robust Optimization* by Ben-Tal, Nemirovski and El Ghaoui.

Robust Optimization

Robust Optimization (RO):

1. Decisions are here-and-now, to be made before data is revealed.
2. Decision maker is responsible for realisations in, and only in, the uncertainty set.
3. Constraints are “hard”, no violations allowed.

Advantages:

- Only crude information (set of possible realisations) needed.
- Computational tractability.

Numerical example

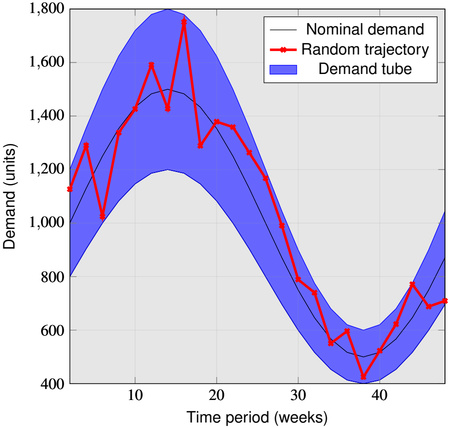
LP model (Ben-Tal et al. (2004))

Minimize production costs over 24 periods

subject to:

- Bounds on production
- Bounds on inventory levels (V_{max} and V_{min})
- All uncertain demand is met

(production costs seasonal)



Adjustable Robust Optimization

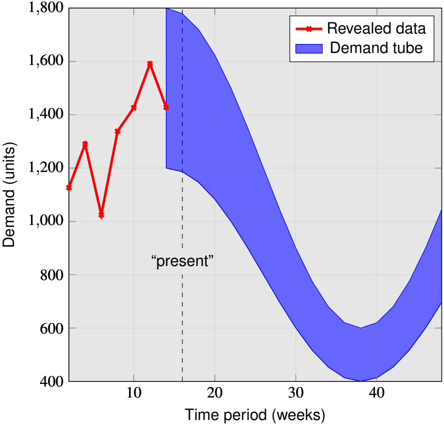
Adjustabe Robust Optimization (ARO) is an extension of RO for multistage optimization problems where some decisions are wait-and-see.

These adjustable decisions are functions of the revealed data from previous periods.

Crucially, the wait-and-see decisions in ARO rely on exact revealed data.

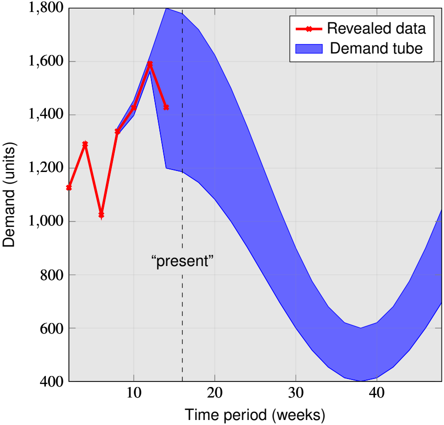
In practice, revealed data is also inexact which may lead to poor performance of ARO...

Numerical example - ARO assumption



Crucially, ARO relies on exact revealed data.

What if revealed data is inexact?



Much evidence that revealed data is inexact! What are the consequences for ARO?

Contributions

1. Reliance on data ‘as is’ may lead to poor performance of ARO if revealed data is inexact.
2. New method with decision rules based on inexact revealed data.
    1. Uses convex analysis (*conjugates and support functions*).
    2. Applicable to many types of convex problems and many different convex uncertainty sets.

Robust counterparts

Uncertain linear constraints of the form:

$$(a + A\zeta)^\top x + d^\top y \leq 0 \qquad \forall \zeta \in \mathcal{Z}$$

- $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ nonadjustable decision variables.
- a the nominal value of the the coefficient for x and $A \in \mathbb{R}^{n \times L}$.
- ζ is the primitive uncertainty residing in a closed convex uncertainty set $\mathcal{Z} \subset \mathbb{R}^L$.
- $d \in \mathbb{R}^m$ is certain.

How to derive equivalent tractable robust counterparts (RC) without '$\forall$' constraints?

Tractable RC

Introduce the indicator function $\delta : \mathbb{R}^L \to \{0, \infty\}$

$$\delta\left(\zeta | \mathcal{Z}\right) = \begin{cases} 0 & \text{if } \zeta \in \mathcal{Z} \\ \infty & \text{if } \zeta \notin \mathcal{Z} \end{cases}$$

and its support function: $\delta^* : \mathbb{R}^L \to \mathbb{R}$

$$\delta^*\left(v | \mathcal{Z}\right) = \max_{\zeta \in \mathcal{Z}} \left\{\zeta^\top v\right\}$$

easy to compute for many $\mathcal{U}_0$!

Uncertainty set $\mathcal{Z}$		$\delta^*(v\|\mathcal{Z})$
box	$\{\zeta : \|\|\zeta\|\|_\infty \leq \theta\}$	$\theta\|\|v\|\|_1$
ball	$\{\zeta : \|\|\zeta\|\|_2 \leq \theta\}$	$\theta\|\|v\|\|_2$
polyhedral	$\{\zeta : b - B\zeta \geq 0\}$	$\min_z \begin{cases} b^\top z & \text{if } B^\top z = v, z \geq 0 \\ \infty & \text{otherwise} \end{cases}$

Tractable RC

Deriving the tractable RC:

$$
\begin{aligned}
(a + A\zeta)^\top x + d^\top y &\leq 0 \qquad \forall \zeta \in \mathcal{Z} \\
&\Leftrightarrow \\
\max_{\zeta \in \mathcal{Z}} \left\{ (a + A\zeta)^\top x \right\} + d^\top y &\leq 0 \\
&\Leftrightarrow \\
a^\top x + d^\top y + \delta^* \left(A^\top x | \mathcal{Z} \right) &\leq 0.
\end{aligned}
$$

See also Ben-Tal, den Hertog and Vial (2014)

Adjustable robust counterpart

Uncertain linear constraints of the form:

$$(a + A\zeta)^\top x + d^\top y(\zeta) \leq 0 \qquad \forall \zeta \in \mathcal{Z}$$

- $x \in \mathbb{R}^n$ nonadjustable and $y(\zeta) \in \mathbb{R}^m$ adjustable.
- a the nominal value of the the coefficient for x and $A \in \mathbb{R}^{n \times L}$.
- $d \in \mathbb{R}^m$ is certain (fixed recourse).
- Linear decision rule based on exact revealed data $y(\zeta) = u + V^\top \zeta$ with $u \in \mathbb{R}^m$ and $V \in \mathbb{R}^{m \times L}$.

Tractable Affinely Adjustable Robust Counterpart (AARC):

$$a^\top x + d^\top u + \delta^* \left(Ax + V^\top d | \mathcal{Z} \right) \leq 0$$

Inexact revealed data in decision rules

Our new methodology deals with uncertain linear constraints of the form:

$$(a + A\zeta)^\top x + d^\top y(\widehat{\zeta}) \leq 0 \qquad \forall \zeta, \widehat{\zeta} \in \mathcal{Z}, \quad (\widehat{\zeta} - \zeta) \in \widehat{\mathcal{Z}}$$

- Affine decision rule based on inexact revealed data $y(\widehat{\zeta}) = u + V\widehat{\zeta}$ with $u \in \mathbb{R}^m$ and $V \in \mathbb{R}^{m \times L}$.
- Estimation error $(\widehat{\zeta} - \zeta)$ resides in closed convex set $\widehat{\mathcal{Z}}$.

Tractable AARC with decision rules based on inexact revealed data (ARCID):

$$a^\top x + d^\top u + \delta^* \left(A^\top x + w | \mathcal{Z}\right) + \delta^* \left(V^\top d - w | \mathcal{Z}\right) + \delta^* \left(w | \widehat{\mathcal{Z}}\right) \leq 0,$$

with $w \in \mathbb{R}^n$ an additional here-and-now decision variable.

Example with polyhedral uncertainty

Consider the following constraint with decision rule $y(\widehat{\zeta}) = u + V\widehat{\zeta}$ based on inexact revealed data:

$$(a + A\zeta)^\top x + d^\top y(\widehat{\zeta}) \leq 0 \qquad \forall \zeta, \widehat{\zeta} \in \mathcal{Z}, \quad (\widehat{\zeta} - \zeta) \in \widehat{\mathcal{Z}},$$

where $\mathcal{Z} = \{\zeta : \; b - B\zeta \geq 0\}$ and $\widehat{\mathcal{Z}} = \{\xi : \; r - R\xi \geq 0\}$ are polyhedral uncertainty sets with given parameters $B, R \in \mathbb{R}^{l \times p}$ and $b, r \in \mathbb{R}^p$.

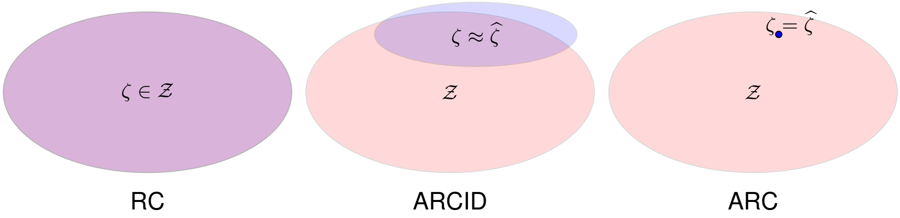
Tractable AARCID:

$$\begin{cases} a^\top x + d^\top u + b^\top (z^1 + z^2) + r^\top z^3 \leq 0 \\ B^\top z^1 = A^\top x + w \\ B^\top z^2 = V^\top d - w \\ R^\top z^3 = w \\ z^1, z^2, z^3 \geq 0, \end{cases}$$

where $w, z^1, z^2, z^3 \in \mathbb{R}^n$ are additional here-and-now variables.

Tractability: LP!

RC, ARC and the new ARCID



Red shaded region: Uncertainty set $\mathcal{Z}$.
Blue shaded region: Estimation uncertainty $\hat{\mathcal{Z}}$.

Large estimation uncertainty $\rightarrow$ ARCID boils down to RC (no extra value of inexact revealed data).
Zero estimation uncertainty $\rightarrow$ ARCID $\equiv$ ARC (revealed data is exact).

In all other situations the new ARCID may outperform both RC and ARC!

Numerical example

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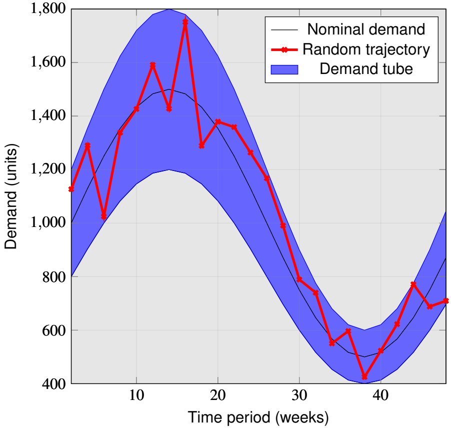

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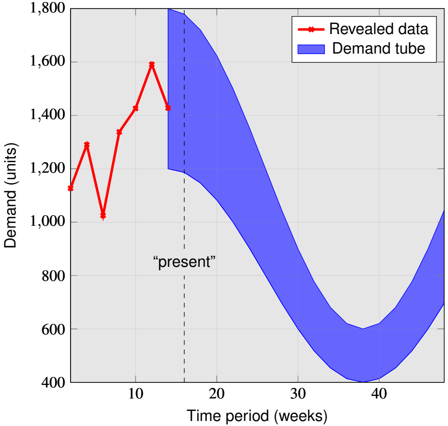

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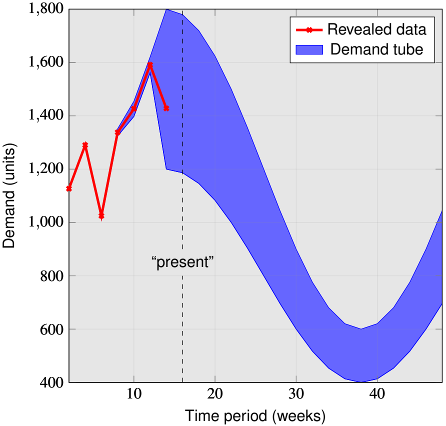


Much evidence that revealed data is inexact! What are the consequences for ARO?

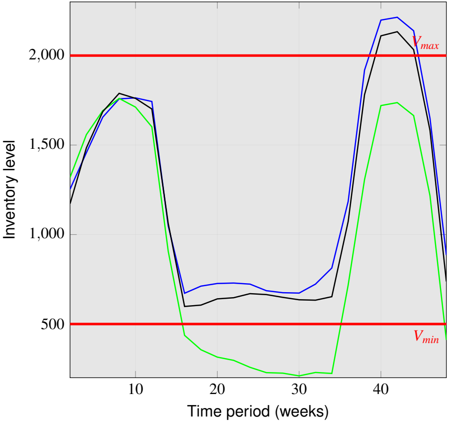
Consequences of inexact revealed data

Option 1: Neglect the inexact nature of the revealed data and use the ARC.

Consider the bounds on inventory levels

All studied cases with inexact revealed data violated these bounds with:

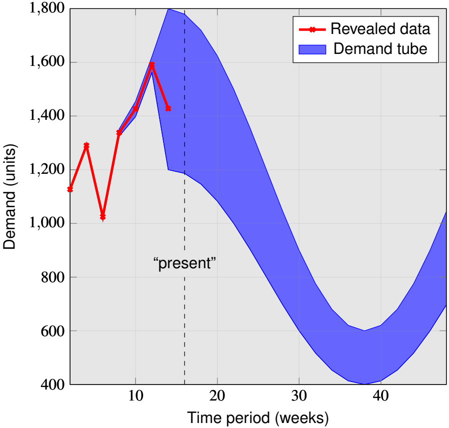
- up to 55 out of 100 cases violate V_{max}.
- up to 80% violates V_{min}.



New ARCID method outperforms ARC

Option 2: Discard the inexact revealed data and only use the exact data in ARC.

- 23 cases, differing in estimation uncertainty, were tested.
- 12 out of 23 cases are infeasible when using the ARC.
- For 9 cases infeasible for ARC one can find feasible solutions with the new ARCID!



Conclusions

- ARC assumes revealed data is exact.
- ARC has two options if revealed data is inexact:
  1. Neglect the inexact nature of revealed data
     $\rightarrow$ Violation of constraints in many cases.
  2. Discard the inexact revealed data in decision rules
     $\rightarrow$ May lead to lower objective value or even infeasibilities.
- New ARCID model is able to use inexact revealed data in the decision rules.
- New ARCID maintains comparable tractability status.